A Proof That the Set of All Algebraic Numbers is Countably Infinite

Chapter Project 2.2

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Introduction

This project develops a method for proving the set of algebraic numbers is countably infinite. The project consists of four parts (a-d) which help construct a method for generating finite sets of polynomials, each of which have a corresponding finite set of solutions. The union of the sets of polynomials is all polynomials; hence the union of the sets of solutions is the set of all numbers that are roots of non-zero polynomials with integer coefficients which, by definition, is the set of algebraic numbers. The final task is to create a mapping of the algebraic number to the set of natural numbers, thus proving that the set of algebraic numbers is countably infinite.

We will present our work on the exercises themselves, followed by a lemma (polynomials of degree n have at most n roots), and finally by the proof itself.

Chapter Project 2.2

PART A

Explain why the set S of all solutions x to the equation, $ax^2 + bx + c = 0$, *where a, b, and c are integers between -5 and 5, is a finite set. Provide a reasonable estimate for the cardinality of the set S.*

By restricting the possible coefficients of the polynomial we are restricting the number of polynomials that can be created. In this case, there are $10 * 11 * 11 = 1210$ second degree polynomials and $10 * 11 = 110$ first degree polynomials. Since polynomials of degree n have at most n roots (see lemma following exercises), this set of polynomials will have at most $(1210 * 2) + 110 = 2530$ solutions. Of course this is a gross overestimation as many of the roots are repeated for various polynomials. For instance $ax^2 = 0$ has the same root for all values of x. Using Mathematica we found that the total number of solutions to this set of polynomials is 916 (the output below, 917, includes an empty set). Thus the cardinality of S is 916.

Mathematica Output:

(Part A*

The number of solutions to polynomials with integer coeffecients between -5 and 5

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Do[S = Union[S, NSolve[(a*x^2) + (b*x) + (c) == 0]], {a, -5,   5}, {b, -5, 5}, {c, -5, 5}];
Print[Length[S]];
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917

So it's clear that by restricting the number of polynomials we are able to create a finite set of solutions to those polynomials.

PART B

If n is a positive integer and if $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x^1 + a_0$ *is a polynomial of degree* $\leq$*n with integer coefficients* $a_n, a_{n-1}, \ldots, a_1, a_0$*, then the height h(p) of p is defined as:*

$$h(p) = n + |a_n| + |a_{n-1}| + \cdots + |a_1| + |a_0|$$

For any positive integer k, define S(k) to be the set of all real solutions x to p(x)=0, where p is a polynomial with integer coefficients that has height k. Find the maximum height of the polynomials described in part a. Compute the cardinality of S(1), S(2), and S(3).

The maximum height of the polynomials of part a would be those of the highest order (2) and with the coefficients of the highest magnitude. If we consider reducible polynomials then the maximum height would be 17 corresponding to the polynomials

$$\pm 5x^2 \pm 5x \pm 5 = 0$$

If, however, we only consider irreducible polynomials then the maximum height would be 16 corresponding to

$$ax^2 + bx + c = 0, where\ a\ and\ b, a\ and\ c, or\ b\ and\ c\ are\ \pm 5\ and\ the\ other\ is\ \pm 4$$

$$S(1) = \{x\ |x\ is\ a\ root\ of\ a\ polynomial\ of\ height\ 1\}$$

$$h(p) = n + |a_n| + |a_{n-1}| + \cdots + |a_1| + |a_0| = 1 \Rightarrow n = 1$$

But then all of the $a_i's$ must be zero, so we have a zero degree polynomial and hence 0 roots.

Cardinality of S(1) = 0

$$S(2) = \{x\ |x\ is\ a\ root\ of\ a\ polynomial\ of\ height\ 2\}$$

$$h(p) = n + |a_n| + |a_{n-1}| + \cdots + |a_1| + |a_0| = 2 \Rightarrow n = \{1,2\}$$

If $n = 2$ then $a_i^{'}s = 0$ so again we have zero roots.

So n must be 1 which means that $a_n = \pm 1$.

Thus the only polynomials of height 2 are:

$$\pm x = 0$$

The root is obviously 0, so the cardinality of S(2) = 1.

$S(3) = \{x \mid x \text{ is a root of a polynomial of height } 3\}$

$h(p) = n + |a_n| + |a_{n-1}| + \cdots + |a_1| + |a_0| = 3 \Rightarrow n = \{1,2,3\}$

If $n = 3$ then $a_i's = 0$ so again we have zero roots.

So n must be 1 or 2.

If n=1 then the possible polynomials of height 3 are:

$x \pm 1 = 0$

$2x = 0$

So the solutions are ± 1 and 0.

If n = 2 then the only possible polynomial of height 3 is:

$x^2 = 0$

So the solution here is 0.

The cardinality of S(3) is 3.

PART C

Explain why S(k) is a finite set for every positive integer k. Show by example that a real number x may belong to S(k) for more than one value of k. Explain why the set A of all algebraic numbers is the union of all of the sets S(k); that is,

$$A = \bigcup_{k=1}^{\infty} S(k)$$

By considering a set P(k) of all polynomials of a certain height, k, we are forming a finite set of polynomials. Thus the set of solutions to these polynomials, S(k), is finite as well as was explained in part a.

In the following example we will show that a real number x may belong to $S(k)$ for more than one value of k.

Polynomial	$h(p)$
$x^2 = 0$	$k = 3$
$2x^2 = 0$	$k = 4$

Both polynomials have the same solution although they have different height.

By letting the height go to infinity we generate all the polynomials at the same time generate a set of all the solutions to all these polynomials. By definition is the set A of all algebraic numbers.

$\bigcup_{k=1}^{\infty} S(k) = A$

PART D

Describe a scheme for "counting" A based on the result of part c. In this way, prove that the set A of all algebraic numbers is countable.

$S(1) = \{\ \ \}$

$S(2) = \{0\}$

$S(3) = \{0,1,-1\}$

$S(4) = \left\{0,1,-1,i,-\frac{1}{2},\frac{1}{2},-2,2\right\}$

$$\bigcup_{k=1}^{\infty} S(k) = A = \{r1, r2, r3, \ldots\}$$

Define $f: A \to \mathbb{N}$ as:

$f(r_1) = 1$

$f(r_2) = 2$

$f(r_3) = 3$

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$f(r_n) = n$

By mapping the algebraic numbers to the naturals we prove that the set A of all algebraic numbers is countable.

Lemma

(1) Lemma: A non-zero polynomial, *p(x),* of degree *n,* $a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x^1 + a_0 = 0$, has at most *n* roots.

Proof

(n = 0): A polynomial of degree zero is a constant so it has 0 roots.

(n = 1): $a_1 x^1 + a_0 = 0 \Leftrightarrow x^1 = -\frac{a_0}{a_1} \Rightarrow a_1 x^1 + a_0$ has 1 root.

Let $k \geq 1$ and assume that a polynomial of degree *k,* $p(k)$ has at most *k* roots. Now consider a polynomial of degree k + 1.

$$p(k+1) \coloneqq \quad a_{k+1}x^{k+1} + a_k x^k + a_{k-1}x^{k-1} + \ldots + a_1 x^1 + a_0 = 0$$

By the Fundamental Theorem of Algebra this polynomial has at least one root, *r.*

Now by the Factor Theorem, (*x* – *r*) is a factor of the polynomial.

So the polynomial $p(k+1)$ can be written as (*x* – *r*)($p(k)$), and $p(k)$has k roots, so $p(k+1)$ has k + 1 roots, which is the degree of the polynomial. ☐

Proof

Define $p(x) := a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x^1 + a_0$, where $a_i \in \mathbb{Z}$ and $p(x)$is non-zero.

Define the height of a polynomial, $h(p)$, as:

$$h(p) = n + |a_n| + |a_{n-1}| + \cdots + |a_1| + |a_0|$$

Let $h(p) = k, k \in \mathbb{Z}^+$.

We wish to show that there are finitely many polynomials, $p(x)$, with height k.

First we will show that all polynomials of height k will be of orders 1 through k-1.

Suppose n > k:

Then $n + |a_n| + |a_{n-1}| + \cdots + |a_1| + |a_0| > k$

$\Leftrightarrow \quad h(p) > k$

But $h(p) = k$, so n $\leq k$.

Suppose n=k:

Then $n + |a_n| + |a_{n-1}| + \cdots + |a_1| + |a_0| = k$

$\Leftrightarrow \quad |a_n| + |a_{n-1}| + \cdots + |a_1| + |a_0| = 0$

$\Rightarrow$ all of the $a_i' s = 0$, but $p(x)$ is non-zero so this isn't possible. Thus n $\neq k$

Therefore n<k.

Suppose n=0 $\Rightarrow p(x) = c$ which is not a polynomial.

Thus for $\forall k \in \mathbb{Z}, 0 < n < k \Rightarrow$ for a given k, $\exists\, p(x)$ of orders 1 through k-1

Now we will show that for a polynomial of a given height, k, and order, n, there are only finitely many possibilities for the values of $a_n, a_{n-1}, \ldots, a_1, a_0$.

$$n + |a_n| + |a_{n-1}| + \cdots + |a_1| + |a_0| = k$$

$$\Leftrightarrow \quad |a_n| + |a_{n-1}| + \cdots + |a_1| + |a_0| = k - n$$

Since $|a_n|, |a_{n-1}|, \ldots, |a_1|, |a_0|$ are all positive there are only finitely many possibilities for their values to sum to k-n.

Therefore, since for a given height k we generate a set of polynomials of a finite number of orders and each of these polynomials have a finite number of possibilities for their coefficients, the set of polynomials of a given height, P(k), is a finite set.

Now let S(k) be the set of all roots of polynomials of height k. By lemma 1, each polynomial of height k will have a finite number of roots, and since there are finitely many polynomials of height k, S(k) is finite as well.

Since every polynomial has a unique height, the $\bigcup_{k=2}^{\infty} P(x)$ is the set of all polynomials. Thus $\bigcup_{k=2}^{\infty} S(k)$ is the set of all roots of all polynomials. Thus, by definition of the algebraic numbers:

$$A = \bigcup_{k=2}^{\infty} S(k)$$

So we have constructed the algebraic numbers, A = {r_1, r_2, r_5, r_4,…}

Now define $f: A \to \mathbb{N}$ as:

$$f(r_1) = 1$$

$$f(r_2) = 2$$

$$f(r_3) = 3$$

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$$f(r_n) = n$$

A function thus defined is a one-to-one correspondence with $\mathbb{N}$, thus A is countably infinite. □